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COMPACT OPERATORS IN BANACH LATTICES

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ABSTRACT
Disjoint sequence methods from the theory of Riesz spaces are used to study compact operators on Banach lattices. A principal new result of the paper is that each positive map from a Banach lattice E to a Banach lattice F with compact majorant is itself compact provided the norms on E' and F are order continuous.

1. Introduction

Compactness criteria for integral operators in various function spaces have been given by a number of authors. Let us cite the work of A. C. Zaanen [29] and T. Andô [1] in Orlicz spaces and of W. A. J. Luxemburg–A. C. Zaanen [13] and J. J. Grobler [9] in Banach function spaces. A related discussion is presented in the book of Krasnoselskii et al. [12] in the Lebesgue spaces. From a more abstract viewpoint, R. J. Nagel and U. Schlotterbeck introduce a class of kernel operators [20] on Banach lattices and, via a representation theorem for such operators, prove a compactness criterion in [21]. Full details of this approach are discussed in the book of H. H. Schaeffer [25].

In the present paper, it is our intention to use the framework of the theory of Riesz spaces (= vector lattices) to provide a systematic approach to the compactness criteria referred to above. Our method is to study the class of linear mappings which map order intervals of a Riesz space E to sets in a Riesz space F which are precompact for the $|\sigma|(F, F^{\sim})$ topology i.e. the topology on F determined by the family of Riesz semi-norms $|g|(|\cdot|)$ for $g \in F^{\sim}$, where $F^{\sim}$ is the Riesz space of order-bounded linear functionals on F. We call such mappings AMAL-compact. It is clear that the class of AMAL-compact maps between Banach lattices contain the compact mappings, and it is equally clear that each AMAL-compact map from an abstract M-space to an abstract L-space is

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compact. In addition, it is not difficult to see directly that each kernel operator in the sense of Luxemburg and Zaanen is AMAL-compact, as are the kernel operators of Nagel and Schlotterbeck, as a glance at the definition in [20] shows readily. The principal result that we give concerning AMAL-compact operators is Theorem 3.4 which asserts that if F is a Dedekind complete Riesz space, and if each order-bounded linear functional on F is a normal integral, then the regular AMAL-compact maps from a Riesz space E to F form a band (= solid, order closed sublattice) in the Dedekind complete Riesz space $L^{\sim}(E;F)$ of regular maps from E to F. Our proof of this fact is based on the spectral theorem of Freudenthal and is completely elementary. As such, it provides the key to avoiding the representation theory of [20] in the further study of compact maps on Banach lattices.

The next task is to characterize those regular AMAL-compact mappings between Banach lattices which are compact. Our characterization is given in Theorem 5.3 and is formulated in terms of disjoint sequences. From another viewpoint, we show how the notions of L-weakly compact and M-weakly compact mappings, introduced by P. Meyer-Nieberg in [18], are related to compactness. Our compactness criterion for regular AMAL-compact operators, together with the fact that the regular AMAL-compact operators form a band containing the band generated by the compact maps, yields the earlier results of [21] and [13] as special cases. Perhaps one application is worth specific mention. We show that each positive map, from a Banach lattice E to the Banach lattice F, with compact majorant is itself compact provided the norms on E' and F are order-continuous. This is stated separately in Theorem 4.5 and provided the stimulus for much of the present paper.

It is natural to ask if the techniques used to study compactness of regular operators may be used in the same setting to discuss the related question of compactness for continuous maps on Banach lattices. To this end, we offer Theorem 5.5 as one characterization of compact maps between Banach lattices. This Theorem is suggested by related results in [12]. It is now convenient to consider the class of continuous maps which map (norm) bounded sets of the Banach lattice E to $|\sigma|(F, F')$ precompact sets in the Banach lattice F, where F' now denotes the (Banach) dual of F. This class of mappings we call PL-compact. Our criterion for compactness is then formulated in terms of PL-compact mappings and disjoint sequences. A major technical tool used in these characterizations is a disjoint sequence theorem given in [3]. In section 2 of the present paper, we present a new result of this type and we remark that our techniques do not require assumptions of some restricted form of topological completeness.

As a final application of our results, we show how to derive certain theorems of Andô [1], Pitt [23] and Rosenthal [24] in the setting of Banach lattices with known indices. We must point out that the results of [24] are formulated for operators defined on subspaces of L^p-spaces rather than on the L^p-spaces themselves, and in this sense are stronger than, it would appear, can be achieved by the techniques to which we have chosen to limit ourselves. Nonetheless, we are inclined to the view that the direct order theoretic arguments of the present paper, which are entirely different to those used in [24], may not be entirely without interest.

The setting for the paper then is the theory of Riesz spaces as outlined, for example, in [7], [15] or [25], and we shall occasionally use results from these sources without specific mention.

Some of the results of this paper were announced at the Oberwolfach meeting on Riesz Spaces and Order-Bounded Operators in June, 1977.

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2. Disjoint sequences and approximately order-bounded sets

In this section, we present some results concerning the behaviour of sequences of mutually disjoint elements in a general Riesz space. The principal result of this section, Theorem 2.5, gives a disjoint sequence characterization of sets which are, in a certain sense, approximately contained in order intervals of a Riesz space. While this result has its origins in the well-known Grothendieck and Dunford–Pettis characterizations of weakly compact sets in abstract L-spaces, we shall use it as a convenient tool in the study of compact operators on Banach lattices.

We begin with a technical result which gives a basis for a new approach to the problem of extracting disjoint sequences. If E is a Riesz space and $0 \leqq w \in E$, we shall denote by E_w the order ideal in E generated by w. If A is a subset of the Riesz space E, we shall write

$$A^d = \{z : z \in E, |z| \wedge |y| = 0 \text{ for all } y \in A\}.$$

LEMMA 2.1. *Let E be a Riesz space. Let $e, u \in E^+$, let $\{f_n\}_{n=1}^{\infty}$, g be positive linear functionals on E and let $\eta > 0$. There exists $v \in [0, u]$ such that $g(u - v) < \eta$ and*

$$f_n(x) = \sup\{f_n(y) \colon y \in [0, x] \cap (E_v + E_v^d)\}$$

for each $n = 1, 2, \cdots$ *and for each* $x \in E_e$.

PROOF. Let $\delta > 0$ be such that $\delta g(e) \leqq \eta$, and for $\alpha \in [0, \delta]$, let $v_\alpha = (u - \alpha e)^+$, let H_α be the ideal in E generated by v_α, and for $n = 1, 2, \cdots$, let

$$\zeta_n(\sigma) = \inf\{f_n(e - y) \colon y \in [0, e] \cap (H_\alpha + H_\alpha^d)\}.$$

We claim that there is a number $\alpha \in]0, \delta]$ such that $\zeta_n(\alpha) = 0$ holds for each $n = 1, 2, \cdots$. Suppose this is not so. There exists a number $\gamma > 0$ and a natural number m such that $\{\alpha \colon \zeta_m(\alpha) \geqq \gamma\}$ is uncountable and so there exists a strictly increasing sequence $\{\alpha_n\}_{n=1}^\infty$ in $]0, \delta]$ such that $\zeta_m(\alpha_n) \geqq \gamma$ for each $n = 1, 2, \cdots$. Let $\delta_n > 0$ be such that $\alpha_{n+1} - \alpha_n \geqq \delta_{n+1} + \delta_n$, so that the intervals $]\alpha_n - \delta_n, \alpha_n + \delta_n]$ are disjoint. Now set

$$y_n = e \wedge \delta_n^{-1}(u - \alpha_n e)^+, \qquad z_n = e \wedge \delta_n^{-1}(u - \alpha_n e)^-,$$

$$u_n = e - (y_n + z_n)$$

so that $y_n \in H_{\alpha_n}$ and $z_n \in H_{\alpha_n}^d$. Now, for $i < j$

$$\begin{aligned}
\delta_i u_i \wedge \delta_j u_j &\leqq \delta_i(e - y_i) \wedge \delta_j(e - z_j) \\
&= (\delta_i e - (u - \alpha_i e)^+)^+ \wedge (\delta_j e - (u - \alpha_j e)^-)^+ \\
&\leqq ((\delta_i + \alpha_i)e - u)^+ \wedge ((\alpha_j - \delta_j)e - u)^- \\
&= 0
\end{aligned}$$

since $\alpha_i + \delta_i \leqq \alpha_j - \delta_j$. So the sequence $\{u_i\}_{i=1}^\infty \subset [0, e]$ is disjoint and

$$\sum_{i=1}^{\infty} f_m(u_i) \leqq f_m(e) < \infty.$$

This is impossible since $f_m(u_i) \geqq \zeta_m(\alpha_i) \geqq \gamma > 0$ holds for $i = 1, 2, \cdots$.

Let now $\alpha \in]0, \delta]$ be such that $\zeta_m(\alpha) = 0$, $m = 1, 2, \cdots$ and set $v = v_\alpha$. Then

$$g(u - v) = g(u \wedge ae) \leqq \delta g(e) \leqq \eta.$$

Suppose now that $x \in E_e^+$. There exists a natural number k such that $0 \leqq x \leqq ke$. Given $\varepsilon > 0$, and given the natural number m, there exist $y_1 \in E_v \cap [0, e]$, $z_1 \in E_v^d \cap [0, e]$ such that

$$kf_m(e - (y_1 + z_1)) < \varepsilon.$$

Setting $y = x \wedge ky_1 \in E_v$ and $z = x \wedge kz_1 \in E_v^d$, we have

$$0 \leqq x-(y+z) \leqq k(e-(y_1+z_1))$$

so that $f_m(x-(y+z)) \leqq \varepsilon$ and the lemma follows.

COROLLARY 2.2. *Let E be a Riesz space, let $e \in E^+$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of positive linear functionals on E. Let $u_0, \cdots, u_p \in E^+$ be disjoint and let $g_0, \cdots, g_p$ be positive linear functionals on E. Given $\eta > 0$, there exist $v_j \in [0, u_j]$ such that*

$$f_n(x) = \sup\{f_n(y) \colon y \in [0, x] \cap (H_0 + \cdots + H_p + H)\}$$

for each $x \in E_e^+$ and each $n = 1, 2, \cdots$, where H_j denotes the ideal generated by v_j and $H = \bigcap_{j \leq p} H_j^d$.

PROOF. The proof is by induction on p. Given $u_0, \cdots, u_p$ choose v_0 by Lemma 2.1 and apply the inductive hypothesis to the shorter sequence $u_1, \cdots, u_p$ and the positive linear functionals f'_n given by

$$f'_n(x) = \sup\{f_n(y) \colon y \in [0, x] \cap H_0^d\}, \ x \in E^+, \ n = 1, 2, \cdots.$$

THEOREM 2.3. *Let E be a Riesz space with an order unit and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of positive linear functionals on E. Let $A \subseteq E$ be a solid set and suppose that $\varepsilon > 0$ is such that for each non-negative integer p there exist disjoint elements $u_0, u_1, \cdots, u_p \in A^+ = A \cap E^+$ such that $\sup_{n \geq p} f_n(u_j) \geqq \varepsilon$ for every $j \leqq p$. For any $\delta < \varepsilon$, there exists a disjoint sequence $\{x_n\}_{n=1}^{\infty}$ in A^+ such that $\limsup_{n \to \infty} f_n(x_n) \geqq \delta$.*

PROOF. Let $0 < \delta'$ satisfy $0 < \delta < \delta' < \varepsilon$. It clearly suffices to show that there exists $0 \leqq z \in A$ such that $\sup_n f_n(z) \geqq \delta'$ and such that, for each natural number p, there exist elements $u_0, \cdots, u_p \in A^+ \cap E_z^d$ such that $\sup_{n \geq p} f_n(u_j) \geqq \delta'$ for each $j \leqq p$. To this end, let the positive integer k be chosen to satisfy

$$k(\varepsilon + \delta') \geqq 2\delta'(k+1)$$

and let $\tilde{u}_0, \cdots, \tilde{u}_k \in A^+$ be disjoint and such that $\sup_n f_n(\tilde{u}_j) \geqq \varepsilon$ for $j \leqq k$. By Corollary 2.2, there exist elements $v_0, \cdots, v_k$ with $v_j \in [0, \tilde{u}_j]$ such that

$$\sup_n f_n(v_j) \geqq \delta', \qquad 0 \leqq j \leqq k,$$

and

$$f_n(x) = \sup\{f_n(y) \colon y \in [0, x] \cap G\}$$

for all $n = 1, 2, \cdots$, and all $x \in G$ where $G = H_0 + \cdots + H_k + (\bigcap_{j \leq k} H_j^d)$, H_j

being the ideal generated by v_j. We now make the following remark. Let $0 \leqq w \in E$, let p be a given positive integer and suppose that $\sup_{n \geqq p} f_n(w) \geqq \varepsilon$. Then, there exists j with $0 \leqq j \leqq k$ and $w' \in \bigcap_{i=j} H_i^d \cap [0, w]$ such that $\sup_{n \geqq p} f_n(w') \geqq \delta'$. In fact, there exist $w_0, \cdots, w_k$, $\tilde{w}$ with $w_i \in H_i \cap [0, w]$, $\tilde{w} \in \bigcap_{i=0}^k H_i^d \cap [0, w]$ such that

$$\sup_{n \geqq p} f_n(w_0 + \cdots + w_k + \tilde{w}) \geqq \tfrac{1}{2}(\delta' + \varepsilon).$$

Set $w^* = w_0 + \cdots + w_k + \tilde{w}$ and observe that

$$kw^* \leqq \sum_{j=0}^{k} \Big(\tilde{w} + \sum_{\substack{i=0 \\ i \neq j}}^{k} w_i \Big).$$

It follows that there exists j with $0 \leqq j \leqq k$ such that

$$\delta' \leqq \frac{k(\delta' + \varepsilon)}{2(k+1)} \leqq \frac{k}{k+1} \sup_{n \geqq p} f_n(w^*)$$

$$\leqq \sup_{n \geqq p} f_n \Big(\tilde{w} + \sum_{\substack{i=0 \\ i \neq j}}^{k} w_i \Big)$$

and so we may take

$$w' = \tilde{w} + \sum_{\substack{i=0 \\ i \neq j}}^{k} w_i.$$

By choosing (finite) disjoint sequences in A of arbitrary length, it now follows from the above remark and the pigeon-hole principle, that for each natural number p, there exists $j(p)$ with $0 \leqq j(p) \leqq k$ and disjoint elements $u_0, \cdots, u_p \in \bigcap_{i=j(p)} H_i^d \cap A^+$ such that $\sup_{n \geqq p} f_n(u_i) \geqq \delta'$ for every $i \leqq p$. It follows that there exists i_0, with $0 \leqq i_0 \leqq k$ such that $j(p) = i_0$ for infinitely many p. We may clearly take $z = v_{i_0}$, and the proof of the Theorem is complete.

If E is a Riesz space, we will denote by $E^\sim$ the order dual of E; see [5].

THEOREM 2.4. *Let E be a Riesz space, let $\{f_n\}_{n=1}^\infty$ be a sequence in $E^\sim$ and assume that each countable subset of E is contained in some principal ideal of E. Let $A \subset E$ be a solid set such that*

(i) $\sup_{x \in A} f_n(x) < \infty$, $n = 1, 2, \cdots$.

(ii) $\lim_{n \to \infty} |f_n|(x) = 0$ *for each* $x \in A$.

(iii) $\lim_{n \to \infty} f_n(x_n) = 0$ *for each disjoint sequence* $\{x_n\}_{n=1}^\infty$ *in* A^+.

Then $\lim_{n \to \infty} \sup_{x \in A} |f_n|(x) = 0$.

PROOF. We observe first that there is no loss in generality in supposing that $f_n \geqq 0$. Indeed, for $n = 1, 2, \cdots$

$$\sup_{x \in A} |f_n|(x) = \sup_{x \in A^+} |f_n(x)|$$

$$= \sup\{f_n(y): x \in A, |y| \leqq x\}$$

$$= \sup_{y \in A} f_n(y) < \infty,$$

and so (i) is satisfied with f_n replaced by $|f_n|$. Again, if $\{x_n\} \subset A^+$ is a disjoint sequence, there exist $\{y_n\} \subset A$ such that

$$|f_n|(x_n) \leqq f_n(y_n) + 2^{-n} \quad \text{and} \quad |y_n| \leqq x_n, \qquad n = 1, 2, \cdots.$$

Since $\{y_n^+\}$, $\{y_n^-\}$ are disjoint sequences in A^+, it follows that $\lim_{n\to\infty} |f_n|(x_n) = 0$. Thus we may replace f_n by $|f_n|$ throughout and accordingly we will assume $f_n \geqq 0$ for the remainder of the proof.

Suppose now that the result is false, so that there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset A^+$ such that $\lim_{n\to\infty} f_n(y_n) > 0$.

By passing to a suitable subsequence, we may assume that there exists $\varepsilon > 0$ for which

$$0 < \varepsilon \leqq f_n(y_n) \quad \text{and} \quad f_n\left(\sum_{i<n} y_i\right) \leqq 2^{-n}\varepsilon, \qquad n = 1, 2, \cdots.$$

For $n = 1, 2, 3, \cdots$, set $\alpha_n = \sup_{x \in A} f_n(x) < \infty$. Let p be a given natural number. Let $k(0) = p$ and choose $k(1), \cdots, k(p)$ such that

$$k(j) > k(j-1), \quad 2^{-k(j)+1}\alpha_{k(j-1)} \leqq \varepsilon, \qquad 1 \leqq j \leqq p.$$

Define, for $0 \leqq j \leqq p$,

$$u_j = \left(y_{k(j)} - 2^{k(j)} \sum_{i<j} y_{k(i)} - \sum_{j<i\leqq p} 2^{-k(i)} y_{k(i)}\right)^+.$$

It is a simple calculation to verify that $u_i \wedge u_j = 0$ if $0 \leqq i < j \leqq p$ and that

$$f_{k(j)}(u_j) \geqq \varepsilon \qquad \text{for } 0 \leqq j \leqq p.$$

By assumption, there exists a principal ideal $I \subset E$ such that $\{y_n\} \subset I \cap A$. We may therefore appeal to Theorem 2.3 and part (iii) of the hypothesis of the present Theorem to obtain a contradiction and to complete the proof of the Theorem.

THEOREM 2.5. *Let E be a Riesz space and let $A \subset E$, $B \subset E^{\sim}$ be solid sets. Suppose that every countable subset of E is included in some principal ideal of E. The following statements are equivalent.*

(i) $\sup_{f \in B} |f(x)| < \infty$ *for every* $x \in A$, $\sup_{x \in A} |f|(x) < \infty$ *for every* $f \in B$ *and* $\lim_{n\to\infty} \sup_{f \in B} |f(x_n)| = 0$ *for every disjoint sequence* $\{x_n\}_{n=1}^{\infty} \subset A^+$.

(ii) *For every* $\varepsilon > 0$, *there exists* $w \in E^+$ *and* $h \in E^{\sim +}$ *such that*

$$(|f|(|x| - w)^+) \leqq \varepsilon, \qquad (|f| - h)^+(|x|) \leqq \varepsilon$$

for all $x \in A$, $f \in B$.

(iii) *Same as* (ii), *but requiring* w, h *to be finite sums of elements in* A^+, B^+ *respectively.*

(iv) $\sup_{f \in B} |f(x)| < \infty$ *for every* $x \in A$, $\sup_{x \in A} |f(x)| < \infty$ *for all* $f \in B$, *and* $\lim_{n\to\infty} \sup_{x \in A} |f_n(x)| = 0$ *for every disjoint sequence* $\{f_n\}_{n=1}^{\infty} \subset B^+$.

PROOF. (i) $\Rightarrow$ (iii). Assume that (i) is true and let $\varepsilon > 0$ be given. We show first that there exists w, a finite sum of elements of A^+, such that

$$|f|(|x| - w)^+ \leqq \varepsilon \qquad \text{for all } f \in B \text{ and } x \in A.$$

Suppose no such w exists. There exist sequences $\{y_n\}_{n=1}^{\infty} \subset A^+$ and $\{f_n\}_{n=1}^{\infty} \subset B^+$ such that

$$f_n\left(\left(y_n - 2^n \sum_{i<n} y_i\right)^+\right) > \varepsilon, \qquad n = 1, 2, \cdots.$$

Set $z_n = (y_n - 2^n \sum_{i<n} y_i)^+$ and define $f'_n \in E^{\sim}$ by

$$f'_n(x) = \sup_k f_n(x \wedge k z_n), \qquad x \in E^+, \quad n = 1, 2, \cdots.$$

It follows that $f'_n((2^n \sum_{i<n} y_i - y_n)^+) = 0$ and so $f'_n(y_i) \leqq 2^{-n} f'_n(y_n)$ holds for $i < n$. Set $\alpha_n = f'_n(y_n)$ and observe that

$$\alpha_n \geqq f_n(z_n) \geqq \varepsilon, \qquad n = 1, 2, \cdots.$$

Define $g_n = \varepsilon \alpha_n f'_n$ and note that $0 \leqq g_n \leqq f_n$ holds for $n = 1, 2, \cdots$ and that $\lim_{n\to\infty} g_n(y_i) = 0$ for each $i = 1, 2, \cdots$. Let now A_1 be the solid hull in E of $\{y_i\}_{i=1}^{\infty}$. By assumption, A_1 is contained in a principal ideal of E and so it follows from (i) and Theorem 2.4 that

$$0 = \lim_{n\to\infty} \sup_{x \in A_1} g_n(x) \geqq \liminf_{n\to\infty} g_n(y_n) = \varepsilon$$

which is a contradiction.

To obtain the second part of condition (iii), we use a result of [7]. Let E_1 be the ideal of E generated by A. The natural map $T: E^{\sim} \to E_1^{\sim}$ given by

$$(Tf)(x) = f(x), \qquad x \in E_1, \quad f \in E^{\sim}$$

is a Riesz homomorphism, for if $x \in E_1^+$ and $f \in E^{\sim}$, then

$$\begin{aligned} |Tf|(x) &= \sup\{(Tf)(y) \colon y \in E_1, |y| \leqq x\} \\ &= \sup\{f(y) \colon y \in E, |y| \leqq x\} \\ &= T(|f|)(x). \end{aligned}$$

Let $B_1 = T[B] \subseteq E_1^{\sim}$ and let F be the ideal in E_1 generated by B_1. If $\{x_n\}_{n=1}^{\infty}$ is any order bounded disjoint sequence in E_1^+ then, by the Riesz decomposition property, it is a finite sum of disjoint sequences in A^+ so that

$$\lim_{n\to\infty} \sup_{g\in B_1} |g(x_n)| = \lim_{n\to\infty} \sup_{f\in B} |f(x_n)| = 0.$$

Also B_1 is $\sigma(E_1^{\sim}, E_1)$-bounded since $\sup_{f\in B} |f(x)| < \infty$ for every $x \in A$. Accordingly, by Lemma 81 H of [7] there exists $g_0 \in F^+$ such that

$$(|g| - g_0)^+(w) \leqq \varepsilon \qquad \text{for all } g \in B_1.$$

Let h be a finite sum of elements of B^+ such that $Th \geqq g_0$. We have then

$$(|f| - h)^+(w) = (|Tf| - Th)^+(w) \leqq \varepsilon \qquad \text{for all } f \in B$$

and so

$$(|f| - h)^+(|x|) \leqq (|f| - h)^+(w) + |f|(|x| - w)^+ \leqq 2\varepsilon$$

for all $f \in B$, $x \in A$ as required.

The implication (iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). First note that if $\varepsilon > 0$ is given and if $w \in E^+$, $h \in E^{\sim +}$ are such that $|f|(|x| - w)^+ \leqq \varepsilon$, $(|f| - h)^+(|x|) \leqq \varepsilon$ for every $x \in A$, $f \in B$, then

$$|f|(|x|) \leqq \varepsilon + |f|(|x| \wedge w) \leqq 2\varepsilon + h(|x| \wedge w)$$

holds for all $x \in A$, $f \in B$. In particular, it follows that $\sup_{f\in B, x\in A} |f|(|x|) < \infty$ and so the first part of (i) is satisfied. Next, if $\{x_n\}_{n=1}^{\infty}$ is a disjoint sequence in A^+, then

$$\sup_{f\in B} |f(x_n)| \leqq 2\varepsilon + h(x_n \wedge w), \qquad n = 1, 2, \cdots.$$

However $\sum_{n=1}^{\infty} h(x_n \wedge w) \leqq h(w) < \infty$ and so

$$\limsup_{n\to\infty} \sup_{f\in B} |f(x_n)| \leqq 2\varepsilon$$

and this proves (i).

(iv) $\Rightarrow$ (i). If $\{x_n\}_{n=1}^\infty$ is any disjoint sequence in A^+ and $\{f_n\}_{n=1}^\infty$ is any sequence in B, define $f'_n \in B^+$ via

$$f'_n(x) = \sup_k |f_n|(x \wedge kx_n), \qquad x \in E^+, \quad n = 1, 2, \cdots.$$

It is easy to see that the sequence $\{f'_n\}_{n=1}^\infty$ is disjoint and so

$$\lim_{n\to\infty} |f_n(x_n)| \leqq \lim_{n\to\infty} |f_n|(x_n) = \lim_{n\to\infty} f'_n(x_n) = 0.$$

As the sequence $\{f_n\}_{n=1}^\infty$ is arbitrary, it follows that $\lim_{n\to\infty} \sup_{f\in B} |f(x_n)| = 0$ as required.

The proof of the implication (iii) $\Rightarrow$ (iv) is almost identical to that of the implication (ii) $\Rightarrow$ (i). By this, the proof of the Theorem is complete.

We remark that the hypotheses of Theorem 2.5 are easily seen to be satisfied if the Riesz space E is a Banach lattice. In this special case, the Theorem coincides essentially with proposition 2.2 of [3]. However, the lack of any assumption of topological completeness in the present Theorem 2.5 introduces the technical difficulty which led to Theorem 2.3.

For ease of reference in later sections, we now gather a number of simple consequences of Theorem 2.5 in the case that E is a Banach lattice. Here E' denotes the (Banach) dual of E.

COROLLARY 2.6. *Let E be a Banach lattice. A sequence $\{x_n\}_{n=1}^\infty$ in E is norm convergent to 0 iff*

(i) $\lim_{n\to\infty} f(|x_n|) = 0$ *for every* $f \in E'$.

(ii) $\lim_{n\to\infty} f_n(x_n) = 0$ *for each norm-bounded disjoint sequence* $\{f_n\}_{n=1}^\infty$ *in* E'^+.

PROOF. It is clear that conditions (i), (ii) are satisfied if the sequence $\{x_n\}$ is norm convergent to 0. Conversely, suppose (i), (ii) hold. Observe that (ii) is equivalent to the apparently stronger condition:

(ii)$'$ $\lim_{n\to\infty} f_n(|x_n|) = 0$ for each norm-bounded disjoint sequence $\{f_n\}_{n=1}^\infty$ in E'^+.

Let $\varepsilon > 0$ be given. By Theorem 2.5, there exists $h \in E'^+$ such that $(|f| - h)^+(|x_k|) < \varepsilon$ for all $f \in E'$ with $\|f\| \leqq 1$ and $k = 1, 2, \cdots$. Thus

$$\|x_n\| = \sup\{f(|x_n|) : f \in E'^+, \|f\| \leqq 1\}$$

$$\leqq \varepsilon + h(x_n), \qquad n = 1, 2, \cdots$$

and the result follows from condition (i).

COROLLARY 2.7. *Let E be a Banach lattice. A sequence $\{f_n\}_{n=1}^{\infty}$ in E' is norm convergent to 0 iff*

(i) $\lim_{n\to\infty} |f_n|(x) = 0$ *for each* $x \in E$.

(ii) $\lim_{n\to\infty} f_n(x_n) = 0$ *for each disjoint norm-bounded sequence* $\{x_n\}_{n=1}^{\infty}$ *in* E^+.

The details of proof are similar to those of Corollary 2.6 and are omitted.

It is well known [25] that the norm on a Banach lattice E is order continuous iff each order-bounded disjoint sequence in E^+ is norm convergent to 0.

COROLLARY 2.8. *Let E be a Banach lattice. The following statements are equivalent.*

(i) *E has order continuous norm.*

(ii) *If $x_0 \in E^+$ and $\varepsilon > 0$, there exists $g \in E'$ such that $(|f| - g)^+(x_0) \leqq \varepsilon$ whenever $f \in E'$ satisfies $\|f\| \leqq 1$.*

COROLLARY 2.9. *Let E be a Banach lattice. The following statements are equivalent.*

(i) *The norm on E' is order continuous.*

(ii) *If $f_0 \in E'^+$ and $\varepsilon > 0$, there is a $y \in E^+$ such that $f_0(|x| - y)^+ \leqq \varepsilon$ whenever $x \in E$ and $\|x\| \leqq 1$.*

(iii) *Each disjoint norm-bounded sequence in E is $\sigma(E, E')$ convergent to 0.*

COROLLARY 2.10. *Let E be a Banach lattice and let $A \subset E$ be a bounded set. The following statements are equivalent.*

(i) $\lim_{n\to\infty} \sup_{x\in A} f_n(|x|) = 0$ *whenever $\{f_n\}_{n=1}^{\infty}$ is a norm-bounded disjoint sequence in E'^+.*

(ii) $\lim_{n\to\infty} \|y_n\| = 0$ *whenever $\{y_n\}_{n=1}^{\infty}$ is a disjoint sequence in the positive part of the solid hull of A.*

(iii) *For any $\varepsilon > 0$, there exist $w \in E^+$, $h \in E'^+$ such that*

$$\|(|x| - w)^+\| \leqq \varepsilon, \qquad (|f| - h)^+(|x|) \leqq \varepsilon$$

for all $x \in A$ and $f \in E'$ with $\|f\| \leqq 1$.

It is to be pointed out that Theorems 2.4, 2.5 continue to hold if the assumption

(i) Each countable subset of E is contained in a principal ideal is replaced by any one of the assumptions:

(ii) E has the principal projection property.

(iii) E is Archimedean and B consists of normal integrals.

(iv) E is almost Dedekind σ-complete and B consists of integrals.†

(v) There is a Hausdorff locally solid, locally convex topology on E for which A is bounded, $B \subset E'$ (the linear topological dual of E) and for which monotone order-bounded Cauchy sequences are convergent.

(vi) There is a locally solid, locally convex topology on E for which A is bounded, $B \subset E'$ and whenever $x \in A^+$ and U is a neighboruhood of 0, there exists $z \in [0, x] \cap U$ and a neighbourhood V of 0 such that $V \cap [0, x] \subset [0, z]$.

Moreover, Corollary 2.2 and Theorem 2.3 also hold under assumptions (ii), (iv). (Of course, for the case of Theorems 2.3, 2.4 and Corollary 2.2, the set B is taken to be given sequence $\{f_n\}$.) The proof of each version of Theorem 2.5 follows the general pattern given above for the order unit case, although under assumptions (ii)–(v), more direct methods are available and it is not necessary to appeal to Theorem 2.1; see for example [3] in which Theorem 2.5 is given under assumption (iv), which also clearly covers the important special case of a Banach lattice. It is appropriate also to point out that the proof of Therorem 2.5 under assumption (v) is essentially the argument in [7], theorem 83 B which is used to discuss weak compactness in the duals of abstract M-spaces.

It is upsetting that Theorems 2.4, 2.5 should appear in such a multiplicity of versions. We have been unable, however, to given a common proof for all versions. It is natural to ask whether Theorems 2.4, 2.5 are valid without any additional assumptions. This is not the case, as is shown by the following example:

EXAMPLE 2.11. Let $\mathbf{N}$, $\mathbf{R}$ denote the natural numbers, real numbers respectively. Let $X = [0,1]^{\mathbf{N}}$ and let $E \subset \mathbf{R}^X$ be the Riesz subspace generated by the coordinate projections $X_i : t \to t(i)$, $i \in \mathbf{N}$, $t \in X$. Define $t_i \in X$ for each $i \in \mathbf{N}$ by taking $t_i(i) = 1$, $t_i(j) = 0$ if $i \neq j$.

Note first, that if $x \in E$ and $i \in \mathbf{N}$, there exists $\alpha \geqq 0$ and a finite subset $J \subset \mathbf{N} \setminus \{i\}$ such that

$$(1) \qquad |x - x(t_i)X_i| \leqq \alpha \sum_{j \in J} X_i.$$

In fact the set of $x \in \mathbf{R}^X$ which satisfies (1) is easily seen to be a Riesz subspace of $\mathbf{R}^X$ which contains each X_i. We note the following important fact about E. If

† E is called almost Dedekind σ-complete if it can be embedded as a super-order-dense Riesz subspace of a Dedekind σ-complete space.

$C \subset E^+$ is any disjoint set, there exists $n \in \mathbf{N}$ such that $x(t_i) = 0$ for every $x \in C$ and for every $i \geqq n$. Suppose this is not so. Since $\{i: x(t_i) \neq 0\}$ is finite for each $x \in E$, there exists an infinite $K \subset \mathbf{N}$ and a disjoint family $\{y_i\}_{i \in k}$ in E^+ such that $y_i(t_i) > 0$ for each $i \in k$. For each $i \in k$, let $\alpha_i \geqq 0$ and $J(i) \subset \mathbf{N} \backslash \{i\}$ be such that

$$| y_i - y_i(t_i) X_i | \leqq \alpha_i \sum_{j \in J(i)} X_j.$$

Take any $i \in k$ and observe that as $J(i)$ is finite, there exists $k \in K / J(i)$ with $k \neq i$ and so $k \notin J(i) \cup J(k)$.

Let $t \in X$ be such that $t(k) = 1$, $0 < t(i) < \alpha_k^{-1} y_k(t_k)$ and $t(y) = 0$ for $j \neq i, k$. Then

$$y_i(t) = y_i(t_i) X_i(t) > 0,$$

$$y_k(t) \geqq y_k(t_k) X_k(t) - \alpha_k X_i(t) > 0$$

and so $y_i \wedge y_k \neq 0$, contrary to hypothesis.

If we take A to be the solid hull of $\{X_i: i \in \mathbf{N}\}$ and define f_n by setting $f_n(x) = x(t_n)$, then $\lim_{n \to \infty} \sup_{m \in \mathbf{N}} f_m(u_n) = 0$ for every disjoint sequence $\{u_n\}_{n \in \mathbf{N}}$ in E^+, $\lim_{m \to \infty} f_m(x) = 0$ for every $x \in E$ but $\lim_{m \to \infty} \sup_{x \in A} f_m(x) \neq 0$. Moreover, it is easily seen that there is no $w \in E^+$ for which $| f_m | (| x | - w)^+ \leqq \frac{1}{2}$ holds for every $m \in \mathbf{N}$ and $x \in A$.

3. AMAL-compact operators

If E is a Riesz space, we will denote by $E_n^\sim$ the band in $E^\sim$ of normal integrals (order-continuous linear functionals). For further details see [7] or [14]. If E, F are Riesz spaces, we will denote by $L^\sim(E; F)$ the space of linear mappings from E to F which are expressible as the difference of positive linear mappings from E to F. If F is Dedekind complete, then $L^\sim(E; F)$ is a Dedekind complete Riesz space ([7], 16 D). The elements of $L^\sim(E; F)$ will be called regular mappings.

We begin with an elementary lemma.

LEMMA 3.1. *Let E, F be Riesz spaces and let F be Dedekind complete. For each $0 \leqq x \in E$ and $T, S \in L^\sim(E; F)$, we have*

$$(| T | \wedge | S |)(x) = \inf \left(\sum_{i=1}^{n} | S | x_i \wedge | T | x_i ;\ n = 1, 2, \cdots, x = \sum_{i=1}^{n} x_i, 0 \leqq x_i \in E \right).$$

The proof of the lemma follows directly from 16 F of [7] and the simple identity

$$|T| \wedge |S| = \tfrac{1}{2}\{|T| + |S| - ||T| - |S||\}.$$

We remark that the collection over which the infimum is taken in the above lemma is downwards directed.

To formulate the main result of this section, we introduce first some notation. Let E be a Riesz space. If $0 \leqq x \in E$, we will denote by E_x the order ideal generated by $x \in E$ and we denote by i_x the injection of E_x into E. If F is a Riesz space and $0 \leqq g \in F^\sim$, write $N_g = \{z \in F: g(|z|) = 0\}$ and denote the quotient map of F onto F/N_g by j_g. We denote by $(F; g)$ the completion of F/N_g with respect to the norm $\|\cdot\|$ induced by the map $z \to g(|z|)$, $z \in F$. We remark that $(F; g)$ is an abstract L-space [7]. Finally, if E is a Riesz space, if F is a Banach lattice, and if $0 \leqq x \in E$, then, without risk of confusion, a linear mapping $S: E_x \to F$ will be called compact if $S([-x, x])$ is a relatively compact subset of F.

THEOREM 3.2. *Let E, F be Riesz spaces and suppose that F is Dedekind complete. Let $0 \leqq x \in E$ and let $0 \leqq g \in F_n^\sim$. Let*

$$G_{x,g} = \{T \in L^\sim(E; F): j_g \circ T \circ i_x \text{ is compact}\}.$$

Then $G_{x,g}$ is a band in $L^\sim(E; F)$.

PROOF. It is clear that $G_{x,g}$ is a linear subspace of $L^\sim(E; F)$. Suppose that $\phi \neq C \subseteq G_{x,g}$ is an upwards directed system and that $C \uparrow T_0$ holds in $L^\sim(E; F)$. We show that $T_0 \in G_{x,g}$. Observe that $\{Tx: T \in C\} \uparrow T_0x$ holds in F. Since $g \in F_n^\sim$, there exists, for each $\varepsilon > 0$, an element $T_\varepsilon \in C$, for which $g(T_0x - T_\varepsilon x) \leqq \varepsilon$ and so, for each $z \in E_x$ with $0 \leqq z \leqq x$,

$$\begin{aligned} \|(j_g \circ T_0)z - (j_g \circ T_\varepsilon)z\| &\leqq \|(j_g \circ T_0 - j_g \circ T_\varepsilon)(x)\| \\ &= g((T_0 - T_\varepsilon)(x)) \\ &\leqq \varepsilon. \end{aligned}$$

Now, by assumption, $j_g \circ T_\varepsilon \circ i_x([-x, x])$ can be covered by finitely many ε-balls. Thus $j_g \circ T_0 \circ i_x([-x, x])$ can be covered by finitely many 2ε-balls and so, as ε is arbitrary, $j_g \circ T_0 \circ i_x([-x, x])$ is totally bounded in $(F; g)$. Thus $T_0 \in G_{x,g}$.

It remains to be proved that $G_{x,g}$ is an order ideal in $L^\sim(E; F)$. To this end, we prove first that if $S, T \in L^\sim(E; F)$ if $|S| \wedge |T| = 0$ and if $S + T \in G_{x,g}$ then S and T both belong to $G_{x,g}$. To this end, let $0 \leqq x \in E$, and let $\varepsilon > 0$ be given. From the fact that g is a normal integral on F and by Lemma 3.1, there exist $0 \leqq x_1, \cdots, x_n \in E$ with $x = \sum_{i=1}^n x_i$ such that

$$g\left(\sum_{i=1}^{n} |S|x_i \wedge |T|x_i\right) < \varepsilon.$$

If $z \in [0, x]$, then $z = z_1 + \cdots + z_n$ with each $z_i \in [0, x_i]$. Since $S + T \in G_{x,g}$, it follows that for $1 \leqq i \leqq n$, there exist finite sets $A_i \subset [0, x_i]$ such that for every $u \in [0, x_i]$ there exists $v \in A_i$ with

$$g(|(S+T)(u-v)|) < \frac{\varepsilon}{n}.$$

In particular, for each z_i, there exists $v_i \in A_i$ with

$$g(|(S+T)(z_i - v_i)|) < \frac{\varepsilon}{n}.$$

It follows that, for $1 \leqq i \leqq n$

$$\begin{aligned} |S(z_i - v_i)| &\leqq |(S+T)(z_i - v_i)| + |S(z_i - v_i)| \wedge |T(z_i - v_i)| \\ &\leqq |(S+T)(z_i - v_i)| + |S|x_i \wedge |T|x_i. \end{aligned}$$

Let now $v = \sum_{i=1}^{n} v_i$. We have

$$\begin{aligned} \|j_g \circ S \circ i_x(z - v)\| &= g(|Sz - Sv|) \\ &\leqq \sum_{i=1}^{n} g(|S(z_i - v_i)|) \\ &\leqq \sum_{i=1}^{n} g(|(S+T)(z_i - v_i)|) + g\left(\sum_{i=1}^{n} |S|x_i \wedge |T|x_i\right) \\ &\leqq 2\varepsilon. \end{aligned}$$

If we now set $A = \{v : v = v_1 + \cdots + v_n,\ v_i \in A_i,\ 1 \leqq i \leqq n\}$, then $A \subset [0, x]$ is a finite set and for each $z \in [0, x]$, there exists $v \in A$ with $\|j_g \circ S \circ i_x(z - v)\| < 2\varepsilon$. As ε is arbitrary, it follows that $S \in G_{x,g}$ and, at once, it follows also that $T \in G_{x,g}$.

It now follows that $G_{x,g}$ is a Riesz subspace of $L^{\sim}(E; F)$; for if $T \in G_{x,g}$ then $T = T^+ + (-T^-)$ and $|T^+| \wedge |-T^-| = 0$, so $T^+ \in G_{x,g}$. Moreover if $0 \leqq T \in G_{x,g}$, and $0 \leqq R \leqq T$ satisfies $R \wedge (T - R) = 0$, then necessarily $R \in G_{x,g}$. It follows from the Freudenthal spectral theorem ([15], 40.2) and the first part of the proof, that if $T \in G_{x,g}$ and if $0 \leqq R \leqq T$, then $R \in G_{x,g}$. The proof of the Theorem is complete.

We wish to remark that the proof of Theorem 3.2 presented above is a simplification, due to A. R. Schep, of an earlier proof given by the present authors.

It is now convenient to make the following definition.

DEFINITION 3.3. Let E, F be Riesz spaces. A linear mapping $T: E \to F$ will be called AMAL-compact iff whenever $0 \leqq x \in E$ and $0 \leqq g \in F$, the bicomposition $j_g \circ T \circ i_x: E_x \to (F; g)$ is compact.

We make first the simple remark that if E is an abstract M-space, and if F is an abstract L-space, then the class of AMAL-compact operators from E to F is precisely the class of compact mappings. Further, if E, F are Banach lattices, then each compact operator from E to F is AMAL-compact. It is to be pointed out that each kernel operator in the sense of Nagel and Schlotterbeck [20] is AMAL-compact and it is not a difficult task to see that each kernel operator in the sense of Luxemberg and Zaanen [13] is AMAL-compact.

With these remarks in mind, we give the following reformulation of Theorem 2.2, which constitutes the principal result of this section.

THEOREM 3.4. *Let E be a Riesz space and let F be a Dedekind complete Riesz space. If $F^{\sim} \subseteq F^{\sim}_{n}$, then the regular* AMAL-*compact mappings from E to F form a band in $L^{\sim}(E:F)$.*

PROOF. Denote by G the linear space of regular AMAL-compact mappings from E to F. In the notation of Theorem 3.2, it is clear that $G = \bigcap\{G_{x,g}: x \in E^{+},\ g \in F^{\sim +}\}$. As each $G_{x,g}$ is a band by Theorem 3.2, it follows that G is a band and the proof is complete.

Of course, the content of the preceding Theorem is most readily seen when E is an abstract M-space and F is an abstract L-space; in this special case, the Theorem asserts that the regular compact mappings from E to F form a band in $L^{\sim}(E:F)$. Our motivation for introducing the notion of AMAL-compact mappings is to reduce the study of compact mappings in more general Banach lattices to that of compact maps of abstract M-spaces to abstract L-spaces. This is very close to the point of view expressed by Andô [1] which is, in turn, related to earlier work of A. C. Zaanen [29].

4. PL-compact sets and operators

For the remainder of the paper we shall restrict attention to Banach lattices. If F is a Banach lattice, we denote by F' the topological dual of F.

Let now F be a Banach lattice. Following [16], we say that a set $A \subseteq F$ is

L-weakly compact if A is norm-bounded and $\|y_n\| \to 0$ as $n \to \infty$ whenever $\{y_n\}_{n=1}^{\infty}$ is a disjoint sequence in the positive part of the solid hull of A. Such sets are characterized by Corollary 2.10. If E and F are Banach lattices, then following [18], a linear mapping $T: E \to F$ will be called L-weakly compact if T maps norm-bounded sets of E to L-weakly compact sets in F. We make a further definition, using the notation of the previous section.

DEFINITION 4.1. Let F be a Banach lattice. A set $A \subseteq F$ will be called PL-compact if $j_g(A)$ is relatively compact in $(F; g)$ whenever $0 \leqq g \in F'$.

If E is another Banach lattice, a linear mapping $T: E \to F$ will be called PL-compact iff $\{Tx : \|x\| \leqq 1\}$ is PL-compact.

We now give a simple criterion for relative compactness of sets in Banach lattices with order continuous norm. (Cf. [12], lemma 1.1 and [16] kor. II.4.)

THEOREM 4.2. *Let F be a Banach lattice and let $A \subseteq F$.*

(a) *If A is L-weakly compact and* PL-*compact, then A is relatively compact.*

(b) *If A is relatively compact, then A is* PL-*compact.*

(c) *If F has order continuous norm, then A is relatively compact iff A is L-weakly compact and* PL-*compact.*

PROOF. (a) Suppose that $A \subseteq F$ is L-weakly compact and PL-compact and let $\varepsilon > 0$ be given. By Corollary 2.10 there exists $0 \leqq g \in F'$ such that

$$(|h| - g)^+(|y|) \leqq \varepsilon \qquad \text{whenever } y \in A \text{ and } \|h\| \leqq 1,$$

since A is L-weakly compact. By PL-compactness of A, $j_g(A) \subseteq (F; g)$ is totally bounded and so there exist $y_0, \cdots, y_n \in A$ such that, for each $y \in A$, there exists $i \leqq n$ such that $\|j_g(y - y_i)\| \leqq \varepsilon$. In this case,

$$\begin{aligned} \|y - y_i\| &= \sup_{\|h\| \leqq 1} h(|y - y_i|) \\ &\leqq g(|y - y_i|) + \sup_{\|h\| \leqq 1} (|h| - g)^+(|y - y_i|) \\ &\leqq \|j_g(y - y_i)\| + \sup_{\|h\| \leqq 1} (|h| - g)^+(|y| + |y_i|) \\ &\leqq \varepsilon + 2\varepsilon \end{aligned}$$

as both y, y_i belong to A. It follows that A can be covered by finitely many 3ε-balls; as ε is arbitrary, it follows that A is totally bounded.

(b) is an immediate consequence of the continuity of the maps j_g, $0 \leqq g \in F'$.

(c) Suppose now that F has order-continuous norm and that $A \subset F$ is relatively compact. From Theorem 2.5 each norm-bounded disjoint sequence in F' is $\sigma(F', F)$ convergent to 0, and therefore converges to 0 uniformly on A. By Corollary 2.10, it follows that A is L-weakly compact. The remaining assertions of (c) are now consequences of parts (a) and (b).

COROLLARY 4.3. *Let E, F be Banach lattices. If F has order-continuous norm, then a linear mapping $T: E \to F$ is compact iff T is L-weakly compact and* PL-*compact.*

We now give some sufficient conditions for the AMAL-compact mappings introduced in the previous section to be PL-compact. We recall first the following notion introduced in [18]. If E, F are Banach lattices, a continuous linear mapping $T: E \to F$ will be called M-weakly compact iff T maps norm-bounded disjoint sequences in E to sequences which converge to 0 in F.

If E and F are Banach lattices and if $T: E \to F$ is a continuous linear map, it is a direct consequence of Theorem 2.5 that $T: E \to F$ is L-weakly compact (M-weakly compact) iff $T': F' \to E'$ is M-weakly compact (L-weakly compact). These results are also proved in [18], satz 3.

THEOREM 4.4. *Let E, F be Banach lattices and let $T: E \to F$ be a continuous linear* AMAL-*compact mapping. If either* (a) *T is M-weakly compact or* (b) *$T \in L^{\sim}(E; F)$ and E' has order continuous norm, then T is* PL-*compact.*

PROOF. We observe first that if $0 \leqq g \in F'$ and $\varepsilon > 0$, there exists $0 \leqq x_0 \in E^+$ such that

$$g(|T(|x| - x_0)^+|) \leqq \varepsilon \qquad \text{whenever } \|x\| \leqq 1$$

for one of the following reasons:

(a) If T is M-weakly compact, then Theorem 2.5 applied to the unit ball of E and the set $\{T'h : h \in F', \|h\| \leqq 1\}$ yields the existence of $0 \leqq x_0 \in E$ with

$$\|T(|x| - x_0)^+\| \leqq \frac{\varepsilon}{\|g\|} \qquad \text{whenever } \|x\| \leqq 1$$

and this is clearly sufficient.

(b) If $T \in L^{\sim}(E; F)$, we may write $T = T_1 - T_2$ with $0 \leqq T_1, T_2$ and set $R = T_1 + T_2$. Consider $f = R'g$. By Corollary 2.9, there exists $0 \leqq x_0 \in E$ such that $f((|x| - x_0)^+) \leqq \varepsilon$ whenever $\|x\| \leqq 1$. Now

$$g(|T(|x| - x_0)^+|) \leqq g(R(|x| - x_0)^+) = f((|x| - x_0)^+) \leqq \varepsilon$$

whenever $\|x\| \leqq 1$.

Thus, under either assumption (a) or (b), there exists $0 \leqq x_0 \in E$ such that

$$\|j_g T(|x| - x_0)^+\| \leqq \varepsilon \qquad \text{whenever } \|x\| \leqq 1.$$

If follows that $j_g \circ T(U^+) \subseteq j_g \circ T([0, x_0]) + \varepsilon V$ where U^+ is the positive part of the unit ball of E and V is the unit ball of $(F; g)$. As $j_g \circ T([0, x_0])$ is totally bounded in $(F; g)$ by assumption it follows that $j_g \circ T(U^+)$, and hence $j_g \circ T(U)$, is totally bounded. As g is arbitrary, T is PL-compact.

We may now state one of the main results of the paper.

THEOREM 4.5. *Let E, F be Banach lattices and let $0 \leqq T: E \to F$ be compact. If E' and F have order continuous norms, then every $S \in [0, T] \subset L^{\sim}(E; F)$ is compact.*

PROOF. By Corollary 4.3, T is L-weakly compact and PL-compact. Let $S: E \to F$ satisfy $0 \leqq S \leqq T$. Clearly, S is L-weakly compact. By Theorem 3.4, S is AMAL-compact and so by Theorem 4.4, S is PL-compact. Again by Corollary 4.3, it follows that S is compact.

We remark that proposition IV, 10.2 of [25] and theorem 2.5.10 of [12] are special cases of the present Theorem 4.5.

It is an immediate consequence of Theorem 4.4 above that if E and F are Banach lattices of which E' has order continuous norm, then the class of regular PL-compact maps from E to F coincides with the class of regular AMAL-compact maps. In view of Theorem 3.4, we may state the following result.

THEOREM 4.6. *Let E, F be Banach lattices and suppose that E', F have order continuous norms. Then the set of regular* PL*-compact maps from E to F is a band in $L^{\sim}(E; F)$.*

The preceding Theorem is not valid if the assumption that E' has order continuous norm is omitted. (See Example 4.14 below.) However we can assert the following variant.

THEOREM 4.7. *Let E, F be Banach lattices. Suppose that F has order-continuous norm. Then the set of regular operators from E to F which map order intervals of E to relatively compact sets in F is a band in $L^{\sim}(E; F)$.*

PROOF. We nee only observe that order bounded PL-compact sets in F are L-weakly compact since F has order continuous norm, and hence relatively compact by Theorem 4.2. Thus the stated class of mappings coincides with the band of AMAL-compact operators from E to F.

We devote the remainder of the section to some examples and some results on PL-compact maps of complementary nature. These will supplement applications to be given in later sections. We begin with some examples of PL-compact sets in some familiar spaces.

EXAMPLE 4.8. (a) In any abstract L-space, a set is PL-compact iff it is relatively compact.

(b) If $1 < p < \infty$, a set in l^p is PL-compact iff it is norm-bounded.

In fact, this is an immediate consequence of the reflexivity of l^p, $1 < p < \infty$, and the fact that weakly convergent sequences in l^p, $1 < p < \infty$, converge uniformly on order intervals of $l^{p'}$, where $1/p' + 1/p = 1$. (See also Theorem 7.3 below.)

(c) If X is a measure space of finite magnitude and $1 < p < \infty$, a set $A \subseteq L^p(X)$ is PL-compact iff A is norm-bounded in $L^p(X)$ and relatively compact in $L^1(X)$.

Indeed, this is a direct consequence of the fact that if $\varepsilon > 0$ is given and $0 \leqq g \in L^q(X)$ where $1/p + 1/q = 1$, then g may be written in the form $g = g_1 + g_2$, where $\|g_1\|_\infty < \infty$ and $\|g_2\|_q \leqq \varepsilon$.

(d) If X is a compact Hausdorff space, and $A \subset C(X)$ is $\|\cdot\|_\infty$-bounded, then the following statements are equivalent.

(i) A is PL-compact.

(ii) Every sequence in A has a subsequence which is pointwise convergent on X.

(iii) Every sequence in A has a subsequence which is $\sigma(C, C')$-Cauchy.

For the proof of this statement, we remark first of all that the implication (iii) $\Rightarrow$ (ii) is obvious and that the implications (ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (i) follow from the Riesz representation theorem for C'. For the implication (i) $\Rightarrow$ (ii), observe that if A is PL-compact, then for any finite Radon measure μ on X and for any sequence $\{x_n\}_{n=1}^\infty$ in A, there is a subsequence $\{y_n\}_{n=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that $\int |y_n - y_{n+1}| \, d\mu \leqq 2^{-n}$ for $n = 1, 2, \cdots$. It follows that any pointwise cluster point of $\{y_n\}_{n=1}^\infty$ is a μ-measurable cluster point of $\{x_n\}_{n=1}^\infty$. It is now a consequence of [2], theorem 2F, (v) $\Rightarrow$ (ii), that every sequence in A has a pointwise convergent subsequence, as required.

We turn now to the question of dual characterizations of PL-compact mappings.

THEOREM 4.9. *If E and F are Banach lattices, a continuous linear operator $T: E \to F$ is* PL-*compact iff T' maps order bounded sets in F' to relatively compact sets in E'.*

PROOF. Let $0 \leqq g \in F'$. By the well-known Schauder Theorem, the mapping $j_g \circ T: E \to (F; g)$ is comapct iff the mapping $T' \circ j_g': (F; g)' \to E'$ is compact. However, the mapping $j_g': (F; g)' \to F_g'$ maps norm-bounded sets in $(F; g)'$ to order bounded sets in F_g' and the result follows.

THEOREM 4.10. *Let E, F be Banach lattices and let $T \in L^{\sim}(E; F)$.*

(a) *If T is* PL-*compact and F'' has order continuous norm, then $T': F' \to E'$ is* PL-*compact.*

(b) *If T' is* PL-*compact and E' has order continuous norm, then T is* PL-*compact.*

PROOF. (a) From Theorem 4.6, $T' \in L^{\sim}(F'; E')$ is AMAL-compact and so PL-compact by order continuity of the norm on E'', by Theorem 4.4 (b).

(b) It suffices to show that T' maps order intervals of F' to relatively compact sets in E', by Theorem 4.6. Since $T' \in L^{\sim}(F'; E')$, T' maps order intervals of F' to order-bounded PL-compact sets in E'. However, order bounded sets of E' are L-weakly compact since E' has order continuous norm, and the result follows from Theorem 3.2 (a).

We now give some simple examples to show that the hypotheses of some of the preceding results may not be omitted.

EXAMPLE 4.11. The identity map $I: c_0 \to c_0$ is PL-compact (see Theorem 7.3 below) but I' and I'' are not and so the assumption that E'' has order continuous norm cannot be omitted from Theorem 4.10 (a).

EXAMPLE 4.12. If we define $T: l^1 \to L^1([0,1])$ by writing $Te_n = \phi_n + \phi_0$, where $\{e_n\}_{n=1}^{\infty}$ is the usual basis of l^1 and ϕ_n is the nth Rademacher function (see [24] chap. IV, ex. 14), then T is not PL-compact. However, if $g \in L^{\infty}([0,1])$ satisfies $\|g\|_{\infty} \leqq 1$, then $T'g \in l^{\infty}$ may be written in the form $T'g = y + z$ where $y \in l^2$ satisfies $\|y\|_2 \leqq 1$ and $z \in l^{\infty}$ is a constant sequence. By the criterion of Example 4.5 (d) (or otherwise) T' is PL-compact, and so the assumption that E' has order-continuous norm cannot be omitted from Theorem 4.10 (b).

EXAMPLE 4.13. If $\{l_n\}_{n=1}^{\infty}$ denotes the usual basis in l^2 and if $\{\phi_n\}_{n=1}^{\infty}$ denotes the Rademacher system in $L^2[0,1]$, define $T: l^2 \to L^2$ by writing $Te_n = \phi_n$, $n = 1, 2, \cdots$, then $T': L^2 \to l^2$ is PL-compact but T is not PL-compact. However, T is not regular.

EXAMPLE 4.14. Denote by $\{l_n\}_{n=1}^{\infty}$ the usual basis in l^1 and by $\{\phi_n\}_{n=1}^{\infty}$ the Rademacher system in $L^2[0,1]$. Define $T: l^1 \to L^2[0,1]$ by setting $Te_n = \phi_n$, $n = 1, 2, \cdots$. Then T is regular and a simple calculation shows that $T^+(e_n) = \phi_n^+$ and that $T^-(e_n) = \phi_n^-$ for $n = 1, 2, \cdots$. Moreover T is not compact, $|T|$ is of rank one and so from $T = T^+ - T^-$, $|T| = T^+ + T^-$ it follows that T^+, T^- are non-compact, as may also be seen directly. It is a consequence of the Schauder Theorem that the operators $(T^+)'$, $(T^-)' : L^2[0,1] \to l^{\infty}$ are non-compact positive operators majorized by the rank one operator $|T|'$. It follows that neither the assumption that E' have order continuous norm, nor the assumption that F have order continuous norm may be omitted in Theorem 4.5.

5. Compact operators

It is the aim of this section to exploit systematically the ideas introduced in earlier sections to give general criteria for compactness of operators in Banach lattices. In so doing, we hope to clarify the relationship of various criteria already present in the literature.

We first make some remarks concerning a condition introduced in [21].

THEOREM 5.1. *Let E, F be Banach lattices. Assume that F has order continuous norm. If $0 \leqq T \in L^{\sim}(E; F)$, then T is M-weakly compact iff the operator norm is order continuous on $[0, T] \subseteq L^{\sim}(E; F)$.*

PROOF. (a) Suppose that T is M-weakly compact, and that $\phi \neq C \downarrow 0$ in $[0, T]$. Let $\varepsilon > 0$ be given. As in the proof of Theorem 4.4 (a), there is an $x_0 \in E^+$ such that $\| T(|x| - x_0)^+ \| \leqq \varepsilon$ whenever $\| x \| \leqq 1$. Now $\{Rx_0 : R \in C\} \downarrow 0$ in F and the norm on F is order-continuous, so there is an $R \in C$ with $\| Rx_0 \| \leqq \varepsilon$. It follows that

$$\| Rx \| \leqq \| R(|x|) \| \leqq \| Rx_0 \| + \| R(|x| - x_0)^+ \| \leqq 2\varepsilon$$

whenever $\| x \| \leqq 1$. Thus $\| R \| \leqq 2\varepsilon$ and it follows that $\inf_{R \in C} \| R \| = 0$, as required.

(b) Suppose now that the norm of $L^{\sim}(E; F)$ is order-continuous on $[0, T]$, and that $\{x_n\}_{n=1}^{\infty}$ is a norm-bounded disjoint sequence in E^+. We may define $T_n \in [0, T]$ for $n = 1, 2, \cdots$, by setting

$$T_n x = \sup_k T(x \wedge kx_n), \qquad x \in E^+$$

(cf. [7], 31 B). It is easily seen that $T_n \wedge T_m = 0$ if $m \neq n$ so that

$$\lim_{n\to\infty} \| Tx_n \| = \lim_{n\to\infty} \| T_n x_n \| \leqq \lim_{n\to\infty} \| T_n \| = 0.$$

It has been noted earlier that if E, F are Banach lattices and if $T\colon E \to F$ is a continuous map, then T is L-weakly compact (M-weakly compact) iff T' is M-weakly compact (L-weakly compact). For the case that T is regular, some additional information may be given.

THEOREM 5.2. *Let E, F be Banach lattices and suppose that E' and F have order-continuous norms. If $T \in L^{\sim}(E;F)$, then the following statements are equivalent.*

(i) *T is L-weakly compact.*
(ii) *T is M-weakly compact.*
(iii) *T' is L-weakly compact.*
(iv) *T' is M-weakly compact.*
(v) $\lim_{n\to\infty} g_n(Tx_n) = 0$ *whenever $\{x_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ are norm-bounded disjoint sequences in E^+ and F'^+ respectively.*

PROOF. The equivalences (i) $\Leftrightarrow$ (iv), (ii) $\Leftrightarrow$ (iii) have been noted above. It is clear that both (ii) and (iv) imply (v). We prove first that (v) $\Rightarrow$ (ii). Indeed suppose (v) is satisfied and that $\{x_n\}_{m=1}^{\infty}$ is a disjoint norm-bounded sequence in E^+. Note first that $x_n \to 0$, $\sigma(E, E')$ by Corollary 2.9 so that $|T| x_n \to 0$, $\sigma(F, F')$ as $n \to \infty$. By Corollary 2.6, it follows from (v) that $\lim_{n\to\infty} \| Tx_n \| = 0$.

The proof that (v) $\Rightarrow$ (iv) is similar to that of the implication (v) $\Rightarrow$ (ii), using Corollaries 2.7, 2.8 instead of Corollaries 2.6, 2.9 and the proof is complete.

We are now in a position to present a criterion for compactness of regular operators.

THEOREM 5.3. *Let E, F be Banach lattices and suppose that E', F have order continuous norms. If $T\colon E \to F$ is regular and* AMAL*-compact, then the following statements are equivalent.*

(i) *T is compact.*
(ii) *T is L-weakly compact.*
(iii) *T is M-weakly compact.*
(iv) $\lim_{n\to\infty} g_n(Tx_n) = 0$ *whenever $\{x_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ are disjoint norm bounded sequences in E^+, F'^+ respectively.*

If in addition $T \geqq 0$, we may add

(v) *The norm of $L^{\sim}(E;F)$ is order continuous on $[0, T]$.*

PROOF. The proof is a synthesis of preceding results.

We have noted earlier that kernel operators in the sense of Luxemburg and Zaanen [13] are AMAL-compact. In this way, the equivalence of statements (i)–(iv) of the preceding Theorem 5.3 provides an extension of theorem 7.3 of [13].

Let now E, F be Banach lattices. In [20], Nagel and Schlotterbeck introduce a class of abstract kernel operators from E to F. These abstract kernel operators are defined to be the set of regular mappings $T: E \to F$ for which all bicompositions $j_g \circ T \circ i_x$, $x \in E^+$, $g \in F'^+$ are nuclear. We note that such mappings are clearly AMAL-compact. Under certain conditions, it is proved in [20] that the set of abstract kernel operators coincide with the band generated in $L^{\sim}(E; F)$ by the finite rank mappings from E to F and that this band consists precisely of those regular maps from E to F which may be represented by kernel operators with kernels defined on the structure spaces of E and F. That this representation theory, while of independent interest, may be avoided in a discussion of compactness is a consequence of Theorem 3.5, which asserts that, provided F has order continuous norm, then each regular map in the band generated by the finite rank maps from E to F is already AMAL-compact. Similarly, each regular map from E to F which lies in the band generated by the regular compact maps from E to F is AMAL-compact. It is to be pointed out in this connection that there are positive compact maps on $L^2([0,1])$ which do not lie in the band generated by the finite rank maps (see [8]). With the above remarks in view, the proof of our next result is clear.

COROLLARY 5.4. *Let E, F be Banach lattices. Let E', F have order continuous norms and suppose that $T: E \to F$ is regular.*

If either

(a) *T belongs to the band generated by the finite rank mappings from E to F; or*

(b) $0 \leqq |T| \leqq |S|$ *for some regular compact operator $S: E \to F$*

then the conditions (i)–(iv) (*respectively* (i)–(v) *if* $T \geqq 0$) *of Theorem* 5.3 *are equivalent.*

We remark that part (a) of the above Corollary 5.4 extends the compactness criterion of Nagel and Schlotterbeck [21], satz [5]. (See also [25], ch. IV.)

We turn now to the question of compactness of continuous, rather than regular, mappings. Our main result in this direction uses the notion of PL-compact rather than AMAL-compact mappings and the interested reader may wish to observe the similarity of the following Theorem to theorems 3.3, 3.4 and 3.5 of [12].

THEOREM 5.5. *Let E, F be Banach lattices such that E', F have order continuous norms. If $T: E \to F$ is a continuous linear mapping, then the following statements are equivalent.*

(i) *T is compact.*

(ii) *T is L-weakly compact and* PL-*compact.*

(iii) *T' is L-weakly compact and* PL-compact.

(iv) *T and T' are both* PL-*compact and* $\lim_{n\to\infty} g_n(Tx_n) = 0$ *whenever* $\{x_n\}_{n=1}^{\infty}$ *and* $\{g_n\}_{n=1}^{\infty}$ *are norm-bounded disjoint sequences in* E^+, F'^+ *respectively.*

PROOF. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow from Corollary 4.3 and Schauder's Theorem. It is clear that (ii) and (iii) together imply (iv) since (iii) implies that T is M-weakly compact.

It remains to prove the implication (iv) $\Rightarrow$ (iii). To this end, suppose that T satisfies (iv) and that $\{x_n\}_{n=1}^{\infty}$ is a norm bounded disjoint sequence in E^+. Observe first that $|Tx_m| \to 0$, $\sigma(F, F')$ as $n \to \infty$. In fact, suppose this is not true. Taking a subsequence if necessary, we may suppose that $0 \leqq g \in F'$ and $\varepsilon > 0$ are such that $g(|Tx_n|) \geqq 3\varepsilon$ for $n = 1, 2, \cdots$. Now T is PL-compact, so the sequence $\{j_g(Tx_n)\}_{n=1}^{\infty}$ is relatively compact in $(F; g)$ and there is a strictly increasing sequence $\{n(i)\}_{i=1}^{\infty}$ such that

$$\lim_{i,j\to\infty} g(|Tx_{n(i)} - Tx_{n(j)}|) = 0.$$

Fix k such that $g(|Tx_{n(i)} - TX_{n(k)}|) \leqq \varepsilon$ for every $i \geqq k$. Let h be such that $|h| \leqq g$ and $h(Tx_{n(k)}) \geqq 2\varepsilon$. Then

$$\begin{aligned} h(Tx_{n(i)}) &\geqq 2\varepsilon - h(Tx_{n(k)} - Tx_{n(i)}) \\ &\geqq 2\varepsilon - g(|Tx_{n(k)} - Tx_{n(i)}|) \\ &\geqq \varepsilon \qquad \text{for all } i \geqq k \end{aligned}$$

and this contradicts the fact that $x_{n(i)} \to 0$, $\sigma(E, E')$ as $i \to \infty$ (as E' has order-continuous norm). Thus it follows that $|Tx_n| \to 0$, $\sigma(F, F')$ as $n \to \infty$. The rest of condition (iv) now implies that $\lim_{n\to\infty} \|Tx_n\| = 0$, by Corollary 2.6. It follows that T is M-weakly compact and so T' is L-weakly compact, as required and the proof is complete.

6. Dunford–Pettis operators

If E and F are Banach lattices and $T: E \to F$ is a continuous linear map, then T will be called a Dunford–Pettis operator if T maps weakly compact sets in E

to (norm) compact sets in F. The main result of this section is that the Dunford–Pettis operators in an abstract L-space coincide with the band of AMAL-compact operators.

We observe first that AMAL-compact operators have some properties reminiscent of weakly compact operators in abstract L-spaces.

THEOREM 6.1. *Let E, F be Banach lattices.*

(a) *Each continuous* AMAL-*compact operator from E to F maps L-weakly compact sets in E to* PL-*compact sets in F.*

(b) *If F has order-continuous norm, then each regular* AMAL-*compact operator from E to F maps L-weakly compact sets in E to relatively compact sets in F.*

PROOF. The proof of (a) is a simple modification of the proof of part (a) of Theorem 4.4 and is accordingly omitted.

(b) Suppose F has order-continuous norm and let T: $E \to F$ be regular and AMAL-compact. It suffices to consider only the case that $T \geqq 0$. Let $A \subset E$ be L-weakly compact and let $\varepsilon > 0$. There exists $0 \leqq x_0 \in E$ and $0 \leqq g \in F'$ such that whenever $x \in A$ and $f \in F'$ with $\|f\| \leqq 1$, it follows that

$$\|(|x| - x_0)^+\| < \frac{\varepsilon}{2\|T\|} \quad \text{and} \quad (|f| - g)^+(Tx_0) \leqq \frac{\varepsilon}{2}$$

(by Corollaries 2.10 and 2.8). Thus

$$\|(|Tx| - Tx_0)^+\| \leqq \|T(|x| - x_0)^+\| < \frac{\varepsilon}{2}$$

and

$$(|f| - g)^+(|Tx|) \leqq \frac{\varepsilon}{2} + (|f| - g)^+(|Tx| - Tx_0)^+ \leqq \varepsilon$$

hold for all $x \in A$, $f \in F'$ with $\|f\| \leqq 1$, and it follows from Corollary 2.10, that $T(A) \subset F$ is L-weakly compact. By part (a) above $T(A)$ is PL-compact and so by Theorem 4.2, $T(A)$ is relatively compact in F and the proof is complete.

If E is an abstract L-space, then each continuous T: $E \to E$ is regular ([7], 26 E). Further, order intervals in E are weakly compact. Consequently each Dunford–Pettis operator in $L(E; E)$ is AMAL-compact, by definition. That each AMAL-compact operator in an abstract L-space is Dunford–Pettis is the content of Theorem 6.1 above. These remarks constitute the first part of the proof of the following assertion.

COROLLARY 6.2. *If E is an abstract L-space, then the Dunford–Pettis operators in E form a band in $L^{\sim}(E;E)$ which contains the weakly compact operators.*

The proof of Corollary 6.2 is, of course, completed by remarking that each weakly compact operator in an abstract L-space is a Dunford–Pettis operator. While this is a well known result, we wish to give a proof which is in the spirit of the present paper. We remark that an entirely analogous proof may be given for abstract M-spaces. For different approaches using Banach lattice techniques, the reader is referred to [28] and [26].

THEOREM 6.3. *Let E be an abstract L-space. Each weakly compact operator in E is a Dunford–Pettis operator.*

PROOF. Let $\{x_n\}_{n=1}^{\infty} \subset E$ satisfy $x_n \to 0$, $\sigma(E, E')$ and let $\{f_n\}_{n=1}^{\infty} \subset E'$ satisfy $f_n \to 0$, $\sigma(E', E'')$. It suffices (see [11], proposition 2) to show that $f_n(x_n) \to 0$ as $n \to \infty$. It is a simple (but not elementary) observation that $|f_n| \to 0$, $\sigma(E', E'')$ since E' is an abstract M-space. Let $\varepsilon > 0$ be given. Since E is an abstract L-space, then by [7], lemma 83 A, there exists $0 \leqq x_0 \in E$ such that $\|(|x_n| - x_0)^+\| < \varepsilon$ for $n = 1, 2, \cdots$. We have then

$$|f_n(x_n)| \leqq |f_n|(x_0) + \varepsilon \sup_n \|f_n\|$$

and the result follows.

We remark finally that Theorem 6.1 (b) above contains theorem 5.9 of [12]; see also [9], theorem 4.5.

7. On theorems of Andô, Pitt and Rosenthal

The results of this section are, in large measure, inspired by the paper [1] of T. Andô, who studied compactness criteria for kernel operators in Orlicz spaces. The main theorem of [1] has subsequently been extended to the setting of Banach function spaces by J. J. Grobler [10] and it is our purpose to show in this section that the results of Andô–Grobler are readily amenable to the techniques of previous sections; at the same time we give some new proofs of some results of H. P. Rosenthal [24] concerning compactness of continuous operators between L^p-spaces.

We recall first some notions concerning indices in Banach lattices, which go back to the work of T. Shimogaki [27]. It is appropriate to point out the related work of B. Maurey [16]. The reader's attention is also drawn to the paper [19] of

P. Meyer-Nieberg, and the references contained therein. We will adopt the following terminology which is given in [6].

DEFINITION 7.1. Let E be a Banach lattice and let $1 \leqq p \leqq \infty$.

(a) E is said to have the l^p-composition property iff whenever $\{x_n\} \subset E^+$ is a disjoint sequence for which the sequence $\{\|x_n\|\} \in l^p$, it follows that $\sup_n \|x_1 + \cdots + x_n\| < \infty$.

(b) E is said to have the l^p-decomposition property iff $\{\|x_n\|\} \in l^p$ whenever $\{x_n\} \subset E^+$ is a disjoint order bounded sequence.

(c) The upper index $\sigma(E)$ is defined by

$$\sigma(E) = \inf\{p \geqq 1: E \text{ has the } l^p \text{ decomposition property}\}.$$

The lower index $s(E)$ is defined by

$$s(E) = \sup\{p \geqq 1: E \text{ has the } l^p \text{ composition property}\}.$$

We now gather for ease of reference a number of results concerning indices that we shall need in the sequel. Let E be a Banach lattice.

(a) $1 \leqq s(E) \leqq \sigma(E) \leqq \infty$. If $1 \leqq p < s(E)$, then E has the l^p-composition property; if $\sigma(E) < q \leqq \infty$, then E has the l^q-decomposition property.

(b) For $1 < p < \infty$, the following statements are equivalent:

(i) E has the l^p-composition property.

(ii) For any disjoint norm-bounded sequence $\{x_n\} \subset E$, there is a continuous linear operator $T: l^p \to E$ such that $Te_n = x_n$, where $\{e_n\}$ is the usual basis of l^p.

(iii) There is a constant $M > 0$ such that

$$\left\|\sum_{i \leqq n} x_i\right\| \leqq M\left\{\sum_{i \leqq n} \|x_i\|^p\right\}^{1/p}$$

for every finite disjoint sequence $\{x_i, 1 \leqq i \leqq n\}$ in E^+.

(c) $s(E)^{-1} + \sigma(E')^{-1} = \sigma(E)^{-1} + s(E')^{-1} = 1$ (with the usual convention that $\infty^{-1} = 0$).

(d) If $\sigma(E) < \infty$, then E has order continuous norm; if $s(E) > 1$, then E' has order continuous norm.

It is in order to make some comments concerning the proofs of statements (a)–(d) preceding. Statement (a) is immediate while (d) is a simple consequence of the standard characterization of order-continuous norms; see [5], theorems 2.3, 2.5. Concerning (b), it is clear that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The implication (i) $\Rightarrow$ (iii) is contained in [19], kor. III.7; we remark that another proof may be obtained by using Lemma 2.1 and the argument of [6], theorem 2.12. It follows from theorem 2.15 of [6] that $s(E')^{-1}+\sigma(E)^{-1}=1$. On the other hand, the proof of theorem 2.16 of [6] combined with (b) above makes it clear that E has the l^p-composition property iff E' has the l^q-decomposition property. It follows simply that $s(E)^{-1}+\sigma(E')^{-1}=1$, and statement (c) above follows.

We now prove a technical result which is the essence of the main result of T. Andô [1]. In this connection, the reader is also referred to lemma 3.1 of [12] and to [10], theorem 4.2.

THEOREM 7.2. *Let E and F be Banach lattices and let $T: E \to F$ be a continuous linear map. If $\sigma(F) < s(E)$, then $\lim_{n\to\infty} g_n(Tx_n)=0$ whenever $\{x_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ are norm-bounded disjoint sequences in E^+ and F'^+ respectively.*

PROOF. Note first that $\sigma(F)<\infty$ and $s(E)>1$ and so E', F have order continuous norms. It follows from Corollaries 2.8 and 2.9, that if $\{x_n\}_{n=1}^{\infty}\subset E$ and $\{g_n\}_{n=1}^{\infty}\subset F'$ are norm-bounded disjoint sequences, then $x_n\to 0$, $\sigma(E,E')$ and $g_n \to 0$, $\sigma(F',F)$ as $n\to\infty$.

Suppose now that the Theorem is false. Using the preceding remarks and passing to a subsequence if necessary, we see that there must be $\varepsilon>0$ and disjoint sequences $\{x_n\}_{n=1}^{\infty}\subset E^+$ and $\{g_n\}_{n=1}^{\infty}\subset F'^+$ such that

$$\|x_n\|=\|g_n\|=1,\quad |g_n(Tx_n)|\geqq \varepsilon>0,\qquad n=1,2,\cdots$$

and

$$|g_n(Tx_i)|\leqq 2^{-n},\quad |g_i(Tx_n)|\leqq 2^{-n},\qquad \text{for } i<n.$$

We may clearly suppose that each $g_n(Tx_n)$ is of the same sign.

Now, setting $p=s(E)$ and $q=s(F')$, we see that

$$\frac{1}{p}+\frac{1}{q}<\sigma(F)^{-1}+q^{-1}=1.$$

It follows that there are $r\leqq p$ and $s<q$ such that $1/r+1/s<1$ and there exist $\{\alpha_i\}_{i=1}^{\infty}\in(l^r)^+$, $\{\beta_i\}_{i=1}^{\infty}\in(l^s)^+$ such that $\Sigma_{i=1}^{\infty}\alpha_i\beta_i=+\infty$. Consider $h_k=\Sigma_{j=1}^{k}\beta_jg_j\in F'$ and $x=\Sigma_{i=1}^{\infty}\alpha_ix_i$, which exists in E since E has the l^r-composition property. It follows that

$$\begin{aligned}|h_k(Tx)| &= \sum_{j=1}^{k}\beta_jg_j\left(\sum_{i=1}^{\infty}\alpha_iTx_i\right)\\ &\geqq \left|\sum_{j=1}^{k}\beta_jg_j(\alpha_jTX_j)\right|-\sum_{j=1}^{k}\sum_{i\neq j}|\alpha_ig_j(Tx_i)|\\ &\geqq \varepsilon\sum_{j=1}^{k}\alpha_j\beta_j-\sum_{i\neq j}\alpha\beta 2^{-\max(i,j)}\end{aligned}$$

(where $\alpha = \sup_i \alpha_i$ and $\beta = \sup_j \beta_j$)

$$\geqq \varepsilon \sum_{j=1}^{k} \alpha_j \beta_j - 2\alpha\beta \sum_{m=1}^{\infty} m 2^{-m}$$

$$\to \infty \qquad \text{as } k \to \infty.$$

However F' has the l^s-composition property so $\lim_{k\to\infty} h_k$ exists in F'. This yields a contradiction and the proof is complete.

We remark first that in view of Theorem 5.3, the main compactness result of Andô–Grobler ([10], theorem 4.2) is a direct consequence of the present Theorem 7.2. The usefulness of Theorem 7.2 is that, in combination with Theorem 5.4, the study of compact operators from E to F may be reduced to questions about operators into abstract L-spaces.

We recall that an Archimedean Riesz space F is called discrete if every non-zero ideal contains an atomic element. See [15], ex. 37.22.

THEOREM 7.3. *Let E, F be Banach lattices. Suppose that E' and F have order continuous norms. If F is discrete, then every continuous linear map from E to F is* PL-*compact.*

PROOF. Let E, F satisfy the given conditions, let $T: E \to F$ be a continuous linear map, and let $0 \leqq g \in F'$. In the notation of section 3, it suffices to show that $j_g \circ T: E \to (F; g)$ is compact. Note first that $j_g \circ T$ maps order intervals of E to relatively weakly compact sets of $(F; g)$, since $(F; g)$ is an abstract L-space. This is an immediate consequence of a well-known theorem of Grothendieck [11] (or see [4]). Since F is discrete and since F has order-continuous norm, it follows that $(F; g)$ is discrete also. Therefore relatively weakly compact sets in $(F; g)$ are relatively compact by [5], theorem 4.7.

Thus $j_g \circ T: E \to (F; g)$ is AMAL-compact. By Theorem 4.4 (a) to show that $j_g \circ T$ is PL-compact, it suffices to show that $j_g \circ T$ is M-weakly compact. Since E' has order continuous norm, disjoint norm-bounded sequences in E are weakly convergent to 0 by Corollary 2.9. Again by [4], theorem 4.7, it follows that $j_g \circ T$ maps disjoint norm-bounded sequences in E to sequences which converge to 0 in $(F: g)$ and so T is M-weakly compact. The proof is complete.

We shall need the following result, whose proof is well known (cf. [22], pp. 236–237). It is also a consequence of theorem A2 of [24], being the special case treated at the bottom of p. 208 of [24].

LEMMA 7.4. *If $2 < p < \infty$ and $T: l^p \to H$ is a continuous linear map, where H*

is an abstract L-space, then $\inf_n \|Te_n\| = 0$, *where* $\{e_n\}_{n=1}^\infty$ *is the usual basis of* l^p.

COROLLARY 7.5. *Let E be a Banach lattice and let H be an abstract L-space. If* $s(E) > 2$, *then every continuous linear operator* $T: E \to H$ *is M-weakly compact.*

PROOF. Suppose that $T: E \to H$ is not M-weakly compact. There is a norm-bounded disjoint sequence $\{x_n\}_{n=1}^\infty \subset E^+$ such that $\inf_n \|Tx_n\| > 0$. As $s(E) > 2$, there is a finite $p > 2$ such that E has the l^p-composition property. Accordingly, there exists a continuous linear map $S: l^p \to E$ for which $Se_n = x_n$, $n = 1, 2, \cdots$ where $\{e_n\}_{n=1}^\infty$ denotes the usual basis in l^p. It follows that $TS: l^p \to H$ is a continuous linear operator such that $\inf_n \|TSe_n\| > 0$, which contradicts Lemma 7.4.

We come now to the principal goal of this section.

THEOREM 7.6. *Suppose that E and F are Banach lattices with* $\sigma(F) < s(E)$. *Suppose either*

(α) E' *and F are both discrete.*

(β) *F is discrete and* $\sigma(F) < 2$.

(γ) E' *is discrete and* $s(E) > 2$.

Then every continuous map $T: E \to F$ *is compact.*

PROOF. The relation $\sigma(F) < s(E)$ implies that $\sigma(E) < \infty$ and $s(E) > 1$ and so E', F have order continuous norms.

(α) We first prove (α) for the special case that $E = l^p$ for some $p < \infty$. Thus suppose F is discrete and let $T: l^p \to F$ be a continuous linear map. By PL-compact. 7.3, T is Pl-compact. At the same time, E' is discrete and E', E'' have order continuous norms since $\sigma(F'') = \sigma(F) < \infty$. Again by Theorem 7.3, T': $F' \to E'$ is PL-compact. By Theorem 7.2 and Theorem 5.5, it follows that T is compact.

To prove case (α) in general, we consider T'. Since T is PL-compact by Theorem 7.3, it suffices, in view of Theorems 5.5 and 7.2, to show that T is L-weakly compact, by Corollary 4.3. Observe first that $\sigma(E') < s(F')$. Let $\{g_n\}_{n=1}^\infty$ be a norm-bounded disjoint sequence in F'^+ and suppose that $\sigma(E') < p < s(F')$. There is a continuous linear map $S: l^p \to E'$ defined by $Se_n = g_n$, where $\{e_n\}_{n=1}^\infty$ is the usual basis of l^p. The map $T'S: l^p \to E'$ is continuous and so compact by what was proved above. Since $\{e_n\} \to 0$ weakly in l^p, it follows that $\|T'g_n\| = \|T'Se_n\| \to 0$ as $n \to \infty$. Thus T' is M-weakly compact and so T is L-weakly compact and the proof of (α) is complete.

(β) We show first that T is L-weakly compact. To this end, let $\{g_n\}_{n=1}^{\infty} \subset F'^{+}$ be a norm-bounded disjoint sequence. By Theorem 7.2, the sequence $\{T'g_n\} \subset E'$ converges to 0 uniformly on each norm-bounded disjoint sequence in E^{+}. Now $s(F') > 2$ and so, for any $x \in E^{+}$, $j_x \circ T'\colon F' \to (E;x)$ is M-weakly compact by Corollary 7.5, as $(E;x)$ is an abstract L-space. It follows that $\{|T'g_n|\}$ is $\sigma(F^*, F)$ convergent to 0 and so $\|T'g_n\| \to 0$ as $n \to \infty$ by Corollary 2.7. Thus T' is M-weakly compact and so T is L-weakly compact. By Theorem 7.3, T is PL-compact and the proof is complete by appealing to Corollary 4.3.

(γ) This case follows from (β) by duality. By this, the proof is complete.

We remark that the condition "E' discrete" of parts (α), (γ) of the preceding Theorem is satisfied if, for example, E is discrete and has order continuous norm.

When E and F are l^p-spaces, the above Theorem 7.7 (α) is given in [22]. A stronger version when E, F are L^p-spaces is given in [24]. The method of proof given in [24] is based on a result of Kadec and Pelczynski concerning weakly convergent sequences in L^p for $2 < p < \infty$. It is obvious that our approach is entirely different and it is our hope that the vector lattice methods of this paper yield a clearer understanding of which properties of L^p-spaces are essential for the result.

The behaviour of the index 2 exhibited in the previous Theorem does not occur if attention is restricted to regular operators. This is the content of the final result presented.

THEOREM 7.7. *Let E, F be Banach lattices. Suppose $\sigma(F) < s(E)$.*

(a) *Every $T \in L^{\sim}(E;F)$ is L-weakly compact and M-weakly compact.*

(b) *The set of compact operators in $L^{\sim}(E;F)$ is a band in $L^{\sim}(E;F)$.*

(c) *If either E' is discrete or if F is discrete then every operator in $L^{\sim}(E;F)$ is compact.*

PROOF.

(a) Follows immediately from Theorems 5.2 and 7.2.

(b) Follows from (a) by Theorems 5.3 and 3.5.

(c) Follows from (a) by Theorems 7.3 and 5.3.

In conclusion, we remark that statement (a) of the preceding Theorem was noted in [18] for the case of L^p-spaces.

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